Multipartite Bell-type inequalities for arbitrary numbers of settings and outcomes per site

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Abstract

We introduce a single general representation incorporating in a unique manner all Bell-type inequalities for a multipartite correlation scenario with an arbitrary number of settings and any spectral type of outcomes at each site. Specifying this general representation for correlation functions, we prove that the form of any correlation Bell-type inequality does not depend on spectral types of outcomes, in particular, on their numbers at different sites, and is determined only by extremal values of outcomes at each site. We also specify the general form of bounds in Bell-type inequalities on joint probabilities. Our approach to the derivation of Bell-type inequalities is universal, concise and can be applied to a multipartite correlation experiment with outcomes of any spectral type, discrete or continuous. We, in particular, prove that, for an N-partite quantum state, possibly, infinite dimensional, admitting the $\underbrace{2 \times \cdots \times 2}_{N}$-setting LHV description, the Mermin–Klyshko inequality holds for any two bounded quantum observables per site, not necessarily dichotomic.

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1. Introduction

A Bell-type inequality represents a tight[1] linear probabilistic constraint on correlation functions or joint probabilities that holds under any multipartite correlation experiment admitting a local hidden variable (LHV) description and may be violated otherwise. Proposed first [1–3] as tests on the probabilistic description of quantum measurements, these inequalities are now

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[1] In the present paper, the term *a tight* LHV constraint means that, in the LHV frame, the bounds established by this constraint cannot be improved. On the difference between the terms *a tight linear LHV constraint* and *an extreme linear LHV constraint*, see the end of section 2.1.

widely used in many quantum information schemes and have been intensively discussed in the literature.

Nevertheless, the most analysed versions [4–18] of Bell-type inequalities refer to either a multipartite case with two settings and two outcomes per site or a bipartite case with small numbers of settings and outcomes and we still know a little about Bell-type inequalities for an arbitrary multipartite correlation experiment. Note, however, that a generalized quantum measurement on even a qubit may have infinitely many outcomes.

In the literature on quantum information, finding Bell-type inequalities for larger numbers of settings and outcomes per site is considered to be a computationally hard problem. This is really the case in the frame of the generally accepted polytope approach [19] where the construction of a complete set of extreme Bell-type inequalities is associated with finding of *all* faces of a highly dimensional polytope. However, many of these faces correspond to *trivial*[2] probabilistic constraints while others can be subdivided into only a few classes, each describing extreme Bell-type inequalities of the same form. It was also shown [17] computationally that increasing numbers of settings and outcomes per site, resulting in the appearance of a huge amount of new faces, leads to only a few (or possibly, no any) new forms of extreme Bell-type inequalities for joint probabilities. Moreover, in the case of an inifinite number of outcomes per site, the polytope approach cannot be, in principle, used for the construction of Bell-type inequalities on joint probabilities of arbitrary events, not necessarily of the product form.

The problem is also complicated by the fact that Bell-type inequalities for correlation functions and Bell-type inequalities for joint probabilities are usually considered separately and a *general* link between the forms of these inequalities in an arbitrary multipartite case has not been analysed in the literature[3].

In the present paper, which is a sequel to[4] [20], we make a step in this direction by introducing a *single* general representation (theorem 1, section 2), incorporating in a unique manner *all* tight linear LHV constraints on either correlation functions or joint probabilities arising under an $S_1 \times \cdots \times S_N$-setting N-partite correlation experiment with outcomes of *any* spectral type, discrete or continuous.

Specifying this general representation for correlation functions, we prove (corollaries 1, 2, section 2.1) that the form of any *correlation* Bell-type inequality *does not depend* on a spectral type of outcomes observed at different sites and is determined only by extremal values of outcomes at each site.

The general form of bounds in the tight linear LHV constraints on joint probabilities is specified by corollaries 3, 4 in section 2.2.

All Bell-type inequalities that have been introduced in the literature [4–18] constitute particular cases of this single general representation. We explicitly demonstrate (section 3) this for: (a) the Clauser–Horne–Shimony–Holt (CHSH) inequality [2] for correlation functions; (b) the Clauser–Horne (CH) inequalities [3] for joint probabilities; (c) the Mermin–Klyshko (MK) inequality [6–8] for correlation functions; (d) the Bell-type inequalities for joint probabilities found computationally by Collins and Gisin [17]; (e) the Zohren–Gill inequality [18] for joint probabilities.

Our approach to the derivation of Bell-type inequalities is universal, concise and allows us to extend the applicability ranges of even the well-known Bell-type inequalities. Applying,

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[2] In the sense that these constraints hold under any multipartite correlation experiment, not necessarily admitting an LHV description.

[3] In a 2×2-setting case, this link was considered by Fine [4] for two outcomes per site and by Masanes [15] for $d \geqslant 2$ discrete outcomes at each site.

[4] In [20], we have consistently formalized the probabilistic description of a multipartite correlation experiment, performed on systems of any nature and with outcomes of any spectral type, discrete or continuous.

for example, this appoach to an N-partite correlation experiment, with two settings and any spectral type of outcomes at each site, we derive the Bell-type inequality (section 3.3) that, being specified for a quantum case, takes the form of the Mermin–Klyshko (MK) inequality [6–8] for spin measurements on N qubits. This proves that, for a quantum state ρ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, admitting the $\underbrace{2 \times \cdots \times 2}_{N}$-setting LHV description, the MK inequality holds for any two bounded quantum observables per site, not necessarily dichotomic. If a Hilbert space $\mathcal{H}_n$, corresponding to the nth site, is infinite dimensional then bounded quantum observables measured at this site may be of any spectral type, discrete or continuous.

2. Linear LHV constraints

Consider an N-partite correlation experiment where an nth party performs $S_n \geqslant 1$ measurements, each specified by a positive integer $s_n \in \{1, \ldots, S_n\}$ and with outcomes $\lambda_n^{(s_n)} \in \Lambda_n^{(s_n)}$ of any spectral type, discrete or continuous, not necessarily real numbers.

This correlation experiment is described by the $S_1 \times \cdots \times S_N$-setting family[5]

$$\mathcal{E} = \{(s_1, \ldots, s_N) \mid s_1 = 1, \ldots, S_1, \ldots, s_N = 1, \ldots, S_N\} \tag{1}$$

of N-partite joint measurements with joint probability distributions

$$P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}\big(\mathrm{d}\lambda_1^{(s_1)} \times \cdots \times \mathrm{d}\lambda_N^{(s_N)}\big), \qquad s_1 = 1, \ldots, S_1, \ldots, s_N = 1, \ldots, S_N, \tag{2}$$

where each distribution $P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}$ may, in general, depend not only on settings of the corresponding joint measurement $(s_1, \ldots, s_N)$ but also on a structure of the whole experiment $\mathcal{E}$.

For an N-partite joint measurement $(s_1, \ldots, s_N) \in \mathcal{E}$, let us denote by[6]

$$\big\langle \Psi\big(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}\big)\big\rangle := \int \Psi\big(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}\big) P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}\big(\mathrm{d}\lambda_1^{(s_1)} \times \cdots \times \mathrm{d}\lambda_N^{(s_N)}\big) \tag{3}$$

the expected (mean) value of a bounded measurable real-valued function Ψ. In particular,

$$\big\langle \phi_1\big(\lambda_1^{(s_1)}\big) \cdot \cdots \cdot \phi_N\big(\lambda_N^{(s_N)}\big)\big\rangle = \int \phi_1\big(\lambda_1^{(s_1)}\big) \cdot \cdots \cdot \phi_N\big(\lambda_N^{(s_N)}\big) P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}\big(\mathrm{d}\lambda_1^{(s_1)} \times \cdots \times \mathrm{d}\lambda_N^{(s_N)}\big) \tag{4}$$

means the expectation of the product of bounded measurable real-valued functions $\phi_1\big(\lambda_1^{(s_1)}\big), \ldots, \phi_N\big(\lambda_N^{(s_N)}\big)$. If outcomes observed at sites: $1 \leqslant n_1 < \cdots < n_M \leqslant N$, are real-valued and bounded then, for any $2 \leqslant M \leqslant N$, the expectation of the product of outcomes observed at these sites, that is,

$$\big\langle \lambda_{n_1}^{(s_{n_1})} \cdot \cdots \cdot \lambda_{n_M}^{(s_{n_M})}\big\rangle = \int \lambda_{n_1}^{(s_{n_1})} \cdot \cdots \cdot \lambda_{n_M}^{(s_{n_M})} P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}\big(\mathrm{d}\lambda_1^{(s_1)} \times \cdots \times \mathrm{d}\lambda_N^{(s_N)}\big), \tag{5}$$

is referred to as *a correlation function*. For $M = N$, this correlation function is called *full*.

If an N-partite joint measurement $(s_1, \ldots, s_N) \in \mathcal{E}$ is *EPR local*[7] then its probability distribution and all marginals of this distribution depend only on settings of the corresponding measurements at the corresponding sites, that is, $P_{(s_1,\ldots,s_N)}^{(\mathcal{E})} \equiv P_{(s_1,\ldots,s_N)}$ and

$$P_{(s_1,\ldots,s_N)}\big(\Lambda_1^{(s_1)} \times \cdots \times \Lambda_{n_1-1}^{(s_{n_1-1})} \times \mathrm{d}\lambda_{n_1}^{(s_{n_1})} \times \cdots \times \mathrm{d}\lambda_{n_M}^{(s_{n_M})} \times \Lambda_{n_M+1}^{(s_{n_M+1})} \times \cdots \times \Lambda_N^{(s_N)}\big)$$
$$\equiv P_{(s_{n_1},\ldots,s_{n_M})}\big(\mathrm{d}\lambda_{n_1}^{(s_{n_1})} \times \cdots \times \mathrm{d}\lambda_{n_M}^{(s_{n_M})}\big), \tag{6}$$

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[5] For details of notation, see sections 2, 3 of [20].

[6] For an integral taken over all values of variables, the domain of integration is not usually specified.

[7] That is, local in the sense meant originally by Einstein, Podolsky and Rosen in [21]. For details, see section 3 of [20].

for any $1 \leqslant n_1 < \cdots < n_M \leqslant N$ and any $1 \leqslant M \leqslant N$. In an EPR local case, the probability distribution of outcomes observed by the nth party under the s_nth measurement depends only on a setting of this measurement and we denote it by

$$P_n^{(s_n)}\big(\mathrm{d}\lambda_n^{(s_n)}\big) := P_{(s_1,\ldots,s_N)}\big(\Lambda_1^{(s_1)} \times \cdots \times \Lambda_{n-1}^{(s_{n-1})} \times \mathrm{d}\lambda_n^{(s_n)} \times \Lambda_{n+1}^{(s_{n+1})} \times \cdots \times \Lambda_N^{(s_N)}\big). \tag{7}$$

The main 'qualitative' statements on a simulation of an $S_1 \times \cdots \times S_N$-setting N-partite correlation experiment in terms of a local hidden variable (LHV) model[8] are introduced in [20]. Below, we specify a single general representation for all linear constraints, on either correlation functions or joint probabilities, arising in the LHV frame. Particular cases of this general representation are further considered in corollaries 1–4.

We stress that the EPR locality does not necessarily imply the existence for a multipartite correlation experiment of an LHV model.

Theorem 1. *Let an $S_1 \times \cdots \times S_N$-setting N-partite correlation experiment (1), with outcomes $\lambda_n^{(s_n)} \in \Lambda_n^{(s_n)}, s_n = 1, \ldots, S_n, n = 1, \ldots, N$, of any spectral type, discrete or continuous, admit an LHV model, conditional or unconditional. Then the tight[9] linear unconditional LHV constraint on expectations:*

$$\inf_{\lambda_1 \in \Lambda_1, \ldots, \lambda_N \in \Lambda_N} \sum_{s_1,\ldots,s_N} \Psi_{(s_1,\ldots,s_N)}\big(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}\big) \leqslant \sum_{s_1,\ldots,s_N} \big\langle \Psi_{(s_1,\ldots,s_N)}\big(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}\big)\big\rangle_{\mathrm{LHV}}$$

$$\leqslant \sup_{\lambda_1 \in \Lambda_1, \ldots, \lambda_N \in \Lambda_N} \sum_{s_1,\ldots,s_N} \Psi_{(s_1,\ldots,s_N)}\big(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}\big) \tag{8}$$

holds for any collection $\big\{\Psi_{(s_1,\ldots,s_N)}\big\}$ of bounded measurable real-valued functions, where $s_1 = 1, \ldots, S_1, \ldots, s_N = 1, \ldots, S_N$, and $\lambda_n := \big(\lambda_n^{(1)}, \ldots, \lambda_n^{(S_n)}\big), \Lambda_n := \Lambda_n^{(1)} \times \cdots \times \Lambda_n^{(S_n)}$.

In particular, the tight linear LHV constraint on product expectations

$$\inf_{\xi_1 \in \Phi_1, \ldots, \xi_N \in \Phi_N} F_N^{(\gamma)}(\xi_1, \ldots, \xi_N) \leqslant \sum_{s_1,\ldots,s_N} \gamma_{(s_1,\ldots,s_N)}\big\langle\phi_1^{(s_1)}\big(\lambda_1^{(s_1)}\big) \cdot \cdots \cdot \phi_N^{(s_N)}\big(\lambda_N^{(s_N)}\big)\big\rangle_{\mathrm{LHV}}$$

$$\leqslant \sup_{\xi_1 \in \Phi_1, \ldots, \xi_N \in \Phi_N} F_N^{(\gamma)}(\xi_1, \ldots, \xi_N) \tag{9}$$

is valid for any bounded measurable real-valued functions $\phi_n^{(s_n)}\big(\lambda_n^{(s_n)}\big), \forall s_n, \forall n$, and any real coefficients $\gamma_{(s_1,\ldots,s_N)}$. Here,

$$F_N^{(\gamma)}(\xi_1, \ldots, \xi_N) = \sum_{s_1,\ldots,s_N} \gamma_{(s_1,\ldots,s_N)}\xi_1^{(s_1)} \cdot \cdots \cdot \xi_N^{(s_N)} \tag{10}$$

is an N-linear form of real vectors

$$\xi_n = \big(\xi_n^{(1)}, \ldots, \xi_n^{(S_n)}\big) \in \mathbb{R}^{S_n}, \qquad n = 1, \ldots, N, \tag{11}$$

and, for any $n \in \{1, \ldots, N\}$,

$$\Phi_n = \big\{\xi_n \in \mathbb{R}^{S_n} \,\big|\, \xi_n^{(s_n)} = \phi_n^{(s_n)}\big(\lambda_n^{(s_n)}\big), \lambda_n^{(s_n)} \in \Lambda_n^{(s_n)}, s_n = 1, \ldots, S_n\big\} \subset \mathbb{R}^{S_n} \tag{12}$$

is the range of the bounded vector-valued function with components $\phi_n^{(s_n)}\big(\lambda_n^{(s_n)}\big)$.

Proof. In view of (3),

$$\sum_{s_1,\ldots,s_N} \big\langle \Psi_{(s_1,\ldots,s_N)}\big(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}\big)\big\rangle$$

$$= \sum_{s_1,\ldots,s_N} \int \Psi_{(s_1,\ldots,s_N)}\big(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}\big) P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}\big(\mathrm{d}\lambda_1^{(s_1)} \times \cdots \times \mathrm{d}\lambda_N^{(s_N)}\big). \tag{13}$$

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[8] For the definition of an LHV model, see section 4 of [20].
[9] The meaning of the term *tight* is specified in footnote 1. On the difference between the terms *tight* and *extreme* with respect to a linear LHV correlation constraint, see the end of section 2.1.

Let family (1) admit an LHV model. Then, by statement (c) of theorem 1 in [20], there exists a joint probability measure

$$\mu_{\mathcal{E}}\big(\mathrm{d}\lambda_1^{(1)} \times \cdots \times \mathrm{d}\lambda_1^{(S_1)} \times \cdots \times \mathrm{d}\lambda_N^{(1)} \times \cdots \times \mathrm{d}\lambda_N^{(S_N)}\big) \tag{14}$$

of all outcomes observed at all sites that returns each distribution $P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}$ of family (1) as the corresponding marginal. Taking this property into account in relation (13), we have

$$\sum_{s_1,\ldots,s_N} \big\langle \Psi_{(s_1,\ldots,s_N)}\big(\lambda_1^{(s_1)},\ldots,\lambda_N^{(s_N)}\big)\big\rangle_{\mathrm{LHV}}$$

$$= \int \left\{ \sum_{s_1,\ldots,s_N} \Psi_{(s_1,\ldots,s_N)}\big(\lambda_1^{(s_1)},\ldots,\lambda_N^{(s_N)}\big) \right\} \; \mu_{\mathcal{E}}(\mathrm{d}\lambda_1 \times \cdots \times \mathrm{d}\lambda_N), \tag{15}$$

where, for short, we denote $\lambda_n = \big(\lambda_n^{(1)},\ldots,\lambda_n^{(S_n)}\big)$ and $\Lambda_n = \Lambda_n^{(1)} \times \cdots \times \Lambda_n^{(S_n)}$. Considering the least upper bound of the second line in (15), we derive

$$\sum_{s_1,\ldots,s_N} \big\langle \Psi_{(s_1,\ldots,s_N)}\big(\lambda_1^{(s_1)},\ldots,\lambda_N^{(s_N)}\big)\big\rangle_{\mathrm{LHV}} \leqslant \sup_{\lambda_1\in\Lambda_1,\ldots,\lambda_N\in\Lambda_N} \sum_{s_1,\ldots,s_N} \Psi_{(s_1,\ldots,s_N)}\big(\lambda_1^{(s_1)},\ldots,\lambda_N^{(s_N)}\big). \tag{16}$$

The left-hand side bound of (8) is proved quite similarly.

In order to prove (9), let us specify (8) with functions $\Psi_{(s_1,\ldots,s_N)}$ of the product form,

$$\Psi_{(s_1,\ldots,s_N)}\big(\lambda_1^{(s_1)},\ldots,\lambda_N^{(s_N)}\big) = \gamma_{(s_1,\ldots,s_N)}\phi_1^{(s_1)}\big(\lambda_1^{(s_1)}\big)\cdot\cdots\cdot\phi_N^{(s_N)}\big(\lambda_N^{(s_N)}\big). \tag{17}$$

For these functions,

$$\sup_{\lambda_1\in\Lambda_1,\ldots,\lambda_N\in\Lambda_N} \sum_{s_1,\ldots,s_N} \Psi_{(s_1,\ldots,s_N)}\big(\lambda_1^{(s_1)},\ldots,\lambda_N^{(s_N)}\big)$$

$$= \sup_{\lambda_1\in\Lambda_1,\ldots,\lambda_N\in\Lambda_N} \sum_{s_1,\ldots,s_N} \gamma_{(s_1,\ldots,s_N)}\phi_1^{(s_1)}\big(\lambda_1^{(s_1)}\big)\cdot\cdots\cdot\phi_N^{(s_N)}\big(\lambda_N^{(s_N)}\big). \tag{18}$$

Denoting $\xi_n^{(s_n)} = \phi_n^{(s_n)}\big(\lambda_n^{(s_n)}\big)$ and taking into account (10)–(12), we have

$$\sup_{\lambda_1\in\Lambda_1,\ldots,\lambda_N\in\Lambda_N} \sum \gamma_{(s_1,\ldots,s_N)}\phi_1^{(s_1)}\big(\lambda_1^{(s_1)}\big)\cdot\cdots\cdot\phi_N^{(s_N)}\big(\lambda_N^{(s_N)}\big) = \sup_{\xi_1\in\Phi_1,\ldots,\xi_N\in\Phi_N} F_N^{(\gamma)}(\xi_1,\ldots,\xi_N).$$

$$\tag{19}$$

The left-hand side of (9) is proved quite similarly.   □

If an $S_1 \times \cdots \times S_N$-setting N-partite correlation experiment (1) admits a *conditional* LHV model then linear combinations of expectations satisfy not only unconditional LHV constraints (8), (9) but also their conditional versions—with the corresponding *conditional* supremums and infimums. The LHV model considered by Bell in [1] represents an example of a conditional LHV model.

Depending on a choice of functions, standing in (9), this constraint reduces to either a general representation for all LHV constraints on correlation functions or a general representation for all LHV constraints on joint probabilities.

2.1. Constraints on correlation functions

Consider an $S_1 \times \cdots \times S_N$-setting N-partite correlation experiment with real-valued outcomes $\lambda_n^{(s_n)} \in \Lambda_n^{(s_n)} \subseteq [-1, 1]$ of any spectral type, discrete or continuous, such that

$$\sup \Lambda_n^{(s_n)} = 1, \qquad \inf \Lambda_n^{(s_n)} = -1, \qquad \forall s_n, \quad \forall n. \tag{20}$$

Note that the description of any multipartite correlation experiment, with at least two outcomes at each site, can be reduced to this case.

For this correlation experiment, let us specify the LHV constraint (9) with functions

$$\phi_n^{(s_n)}\big(\lambda_n^{(s_n)}\big) = \lambda_n^{(s_n)} + z_n^{(s_n)}, \qquad \forall\, s_n, \quad \forall\, n, \tag{21}$$

where each $z_n^{(s_n)}$ is an arbitrary real number. We derive:

$$\sum_{s_1,\dots,s_N} \gamma_{(s_1,\dots,s_N)} \big\langle \phi_1^{(s_1)}\big(\lambda_1^{(s_1)}\big) \cdot \dots \cdot \phi_N^{(s_N)}\big(\lambda_N^{(s_N)}\big) \big\rangle = \sum_{s_1,\dots,s_N} \gamma_{(s_1,\dots,s_N)} z_1^{(s_1)} \cdot \dots \cdot z_N^{(s_N)}$$
$$+ \sum_{\substack{1 \leqslant n_1 < \dots < n_M \leqslant N, \\ M=1,\dots,N}} \sum_{s_{n_1},\dots,s_{n_M}} \gamma_{(s_{n_1},\dots,s_{n_M})} \big\langle \lambda_{n_1}^{(s_{n_1})} \cdot \dots \cdot \lambda_{n_M}^{(s_{n_M})} \big\rangle, \tag{22}$$

where[10]

$$\gamma_{(s_{n_1},\dots,s_{n_M})} := \gamma_{(s_1,\dots,s_N)} \delta_{M,N} + (1 - \delta_{M,N}) \sum_{s_n,\,\forall\, n \neq n_1,\dots,n_M} \left\{ \gamma_{(s_1,\dots,s_n,\dots,s_N)} \prod_{n \neq n_1,\dots,n_M} z_n^{(s_n)} \right\}. \tag{23}$$

For the N-linear form (10), consider the change of variables: $\xi_n = \eta_n + z_n, \forall\, n \in \{1,\dots,N\}$, where $z_n := (z_n^{(1)},\dots,z_n^{(S_n)}) \in \mathbb{R}^{S_n}$ is the real vector with components given by real numbers in (21). We have

$$F_N^{(\gamma)}(\xi_1,\dots,\xi_N) = \sum_{s_1,\dots,s_N} \gamma_{(s_1,\dots,s_N)} z_1^{(s_1)} \cdot \dots \cdot z_N^{(s_N)}$$
$$+ \sum_{\substack{1 \leqslant n_1 < \dots < n_M \leqslant N, \\ M=1,\dots,N}} \sum_{s_{n_1},\dots,s_{n_M}} \gamma_{(s_{n_1},\dots,s_{n_M})} \eta_{n_1}^{(s_{n_1})} \cdot \dots \cdot \eta_{n_M}^{(s_{n_M})}. \tag{24}$$

From (12), (21) it follows that

$$\xi_n \in \Phi_n \quad \Leftrightarrow \quad \eta_n \in \Lambda_n = \Lambda_n^{(1)} \times \dots \times \Lambda_n^{(S_n)} \subseteq [-1,1]^{S_n}, \tag{25}$$

where, due to (20), closure $\overline{\Lambda}_n$ of the bounded set Λ_n satisfies the relation $\{-1,1\}^{S_n} \subseteq \overline{\Lambda}_n \subseteq [-1,1]^{S_n}$.

Substituting (22), (24) into (9) and taking into account (25), we derive

$$\inf_{\eta_1 \in \Lambda_1,\dots,\eta_N \in \Lambda_N} \sum_{\substack{1 \leqslant n_1 < \dots < n_M \leqslant N, \\ M=1,\dots,N}} F_M^{(\gamma)}\big(\eta_{n_1},\dots,\eta_{n_M}\big)$$
$$\leqslant \sum_{\substack{1 \leqslant n_1 < \dots < n_M \leqslant N, \\ M=1,\dots,N}} \sum_{s_{n_1},\dots,s_{n_M}} \gamma_{(s_{n_1},\dots,s_{n_M})} \big\langle \lambda_{n_1}^{(s_{n_1})} \cdot \dots \cdot \lambda_{n_M}^{(s_{n_M})} \big\rangle_{\mathrm{LHV}}$$
$$\leqslant \sup_{\eta_1 \in \Lambda_1,\dots,\eta_N \in \Lambda_N} \sum_{\substack{1 \leqslant n_1 < \dots < n_M \leqslant N, \\ M=1,\dots,N}} F_M^{(\gamma)}\big(\eta_{n_1},\dots,\eta_{n_M}\big), \tag{26}$$

where

$$F_M^{(\gamma)}\big(\eta_{n_1},\dots,\eta_{n_M}\big) = \sum_{s_{n_1},\dots,s_{n_M}} \gamma_{(s_{n_1},\dots,s_{n_M})} \eta_{n_1}^{(s_{n_1})} \cdot \dots \cdot \eta_{n_M}^{(s_{n_M})} \tag{27}$$

is an M-linear form of real vectors

$$\eta_1 = \big(\eta_1^{(1)},\dots,\eta_1^{(S_1)}\big) \in \mathbb{R}^{S_1},\dots,\eta_N = \big(\eta_N^{(1)},\dots,\eta_N^{(S_N)}\big) \in \mathbb{R}^{S_N}.$$

For a further simplification of constraint (26), we need the following property proved in the appendix.

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[10] Here, $\delta_{M,N} = 1$ if $M = N$ and $\delta_{M,N} = 0$ if $M \neq N$.

Lemma 1. *Let, for each bounded set* $\Lambda_n \subseteq [-1, 1]^{S_n}$, $n \in \{1, \ldots, N\}$, *its closure* $\overline{\Lambda}_n$ *satisfy the relation,*

$$\{-1, 1\}^{S_n} \subseteq \overline{\Lambda}_n \subseteq [-1, 1]^{S_n}, \qquad \forall n = 1, \ldots, N. \tag{28}$$

Then

$$\sup_{\eta_1 \in \Lambda_1, \ldots, \eta_N \in \Lambda_N} \sum_{\substack{1 \leqslant n_1 < \cdots < n_M \leqslant N, \\ M=1,\ldots,N}} F_M^{(\gamma)}(\eta_{n_1}, \ldots, \eta_{n_M})$$

$$= \max_{\eta_1 \in \{-1,1\}^{S_1}, \ldots, \eta_N \in \{-1,1\}^{S_N}} \sum_{\substack{1 \leqslant n_1 < \cdots < n_M \leqslant N, \\ M=1,\ldots,N}} F_M^{(\gamma)}(\eta_{n_1}, \ldots, \eta_{n_M}), \tag{29}$$

with a similar expression for infimum.

Substituting (29) into constraint (26), we derive the following corollary of theorem 1.

Corollary 1. *Let an* $S_1 \times \cdots \times S_N$*-setting N-partite correlation experiment (1), with real-valued outcomes*

$$\lambda_n^{(s_n)} \in \Lambda_n^{(s_n)} \subseteq [-1, 1], \qquad \sup \Lambda_n^{(s_n)} = 1, \qquad \inf \Lambda_n^{(s_n)} = -1, \quad \forall s_n, \quad \forall n, \tag{30}$$

of any spectral type, discrete or continuous, admit an LHV model. Then the tight linear LHV constraint on correlation functions,

$$\min_{\eta_1 \in \{-1,1\}^{S_1}, \ldots, \eta_N \in \{-1,1\}^{S_N}} \sum_{\substack{1 \leqslant n_1 < \cdots < n_M \leqslant N, \\ M=1,\ldots,N}} F_M^{(\gamma)}(\eta_{n_1}, \ldots, \eta_{n_M})$$

$$\leqslant \sum_{\substack{1 \leqslant n_1 < \cdots < n_M \leqslant N, \\ M=1,\ldots,N}} \sum_{s_{n_1}, \ldots, s_{n_M}} \gamma_{(s_{n_1}, \ldots, s_{n_M})} \langle \lambda_{n_1}^{(s_{n_1})} \cdot \ldots \cdot \lambda_{n_M}^{(s_{n_M})} \rangle_{\mathrm{LHV}}$$

$$\leqslant \max_{\eta_1 \in \{-1,1\}^{S_1}, \ldots, \eta_N \in \{-1,1\}^{S_N}} \sum_{\substack{1 \leqslant n_1 < \cdots < n_M \leqslant N, \\ M=1,\ldots,N}} F_M^{(\gamma)}(\eta_{n_1}, \ldots, \eta_{n_M}), \tag{31}$$

holds for any collection $\{\gamma_{(s_{n_1}, \ldots, s_{n_M})}\}$ *of real coefficients. Here,* $F_M^{(\gamma)}$ *is an M-linear form defined by (27) and extremums are taken over all* $2^{S_1 + \cdots + S_N}$ *vertices of hypercube* $[-1, 1]^{S_1 + \cdots + S_N} \subset \mathbb{R}^{S_1 + \cdots + S_N}$.

From the definition of a Bell-type inequality, given in the introduction, and corollary 1 it follows that *the form of any correlation Bell-type inequality does not depend on a spectral type of outcomes observed at each site, in particular, on their number and is determined only by extremal values of these outcomes.*

If, in particular, $\gamma_{(s_{n_1}, \ldots, s_{n_M})} = \delta_{N,M} \gamma_{(s_1, \ldots, s_N)}$, then (31) reduces to the tight linear LHV constraint on the full correlation functions,

$$\min_{(\eta_1, \ldots, \eta_N) \in \{-1,1\}^d} F_N^{(\gamma)}(\eta_1, \ldots, \eta_N) \leqslant \sum_{s_1, \ldots, s_N} \gamma_{(s_1, \ldots, s_N)} \langle \lambda_1^{(s_1)} \cdot \ldots \cdot \lambda_N^{(s_N)} \rangle_{\mathrm{LHV}}$$

$$\leqslant \max_{(\eta_1, \ldots, \eta_N) \in \{-1,1\}^d} F_N^{(\gamma)}(\eta_1, \ldots, \eta_N), \tag{32}$$

where $d := S_1 + \cdots + S_N$. Noting that

$$F_N^{(\gamma)}(\eta_1, \ldots, \eta_n, \ldots, \eta_N) = -F_N^{(\gamma)}(\eta_1, \ldots, -\eta_n, \ldots, \eta_N), \tag{33}$$

and points

$$(\eta_1, \ldots, \eta_n, \ldots, \eta_N) \in \mathbb{R}^d, \qquad (\eta_1, \ldots, -\eta_n, \ldots, \eta_N) \in \mathbb{R}^d \tag{34}$$

belong to hypercube $[-1, 1]^d \subset \mathbb{R}^d$ simultaneously, we derive

$$- \min_{(\eta_1, \ldots, \eta_N) \in \{-1, 1\}^d} F_N^{(\gamma)}(\eta_1, \ldots, \eta_N) = \max_{(\eta_1, \ldots, \eta_N) \in \{-1, 1\}^d} F_N^{(\gamma)}(\eta_1, \ldots, \eta_N)$$

$$= \max_{(\eta_1, \ldots, \eta_N) \in \{-1, 1\}^d} \left| F_N^{(\gamma)}(\eta_1, \ldots, \eta_N) \right|. \quad (35)$$

Substituting (35) into (32), we come to the following corollary of theorem 1.

Corollary 2. *Let an $S_1 \times \cdots \times S_N$-setting N-partite correlation experiment (1), with real-valued outcomes $\lambda_n^{(s_n)} \in \Lambda_n^{(s_n)} \subseteq [-1, 1]$, $\sup \Lambda_n^{(s_n)} = 1$, $\inf \Lambda_n^{(s_n)} = -1$, $\forall s_n$, $\forall n$, of any spectral type, discrete or continuous, admit an LHV model. Then the full correlation functions satisfy the tight linear LHV constraint*

$$\left| \sum_{s_1, \ldots, s_N} \gamma_{(s_1, \ldots, s_N)} \langle \lambda_1^{(s_1)} \cdot \ldots \cdot \lambda_N^{(s_N)} \rangle_{\mathrm{LHV}} \right| \leqslant \max_{\substack{\eta_1 \in \{-1, 1\}^{S_1}, \ldots, \\ \eta_N \in \{-1, 1\}^{S_N}}} \left| F_N^{(\gamma)}(\eta_1, \ldots, \eta_N) \right|, \quad (36)$$

for any real coefficients $\gamma_{(s_1, \ldots, s_N)}$.

If a correlation experiment admits a conditional LHV model then, in addition to (31), (36), the correlation functions satisfy also the conditional versions of these constraints—with the corresponding conditional extremums. The original Bell inequality, derived by Bell in [1] in the frame of the conditional LHV model, represents an example of a conditional LHV constraint on the full correlation functions.

We stress that, in corollaries 1, 2, the term *a tight linear LHV constraint* does not mean *an extreme linear LHV constraint.* The difference between these two terms is clearly seen due to the geometric interpretation of, say, constraint (36) in terms of the polytope approach [19].

Namely, for any choice of coefficients $\gamma_{(s_1, \ldots, s_N)}$ in constraint (36) represented otherwise as

$$- \max_{(\eta_1, \ldots, \eta_N) \in \{-1, 1\}^d} \left| F_N^{(\gamma)}(\eta_1, \ldots, \eta_N) \right| \leqslant \sum_{s_1, \ldots, s_N} \gamma_{(s_1, \ldots, s_N)} \langle \lambda_1^{(s_1)} \cdot \ldots \cdot \lambda_N^{(s_N)} \rangle_{\mathrm{LHV}}$$

$$\leqslant \max_{(\eta_1, \ldots, \eta_N) \in \{-1, 1\}^d} \left| F_N^{(\gamma)}(\eta_1, \ldots, \eta_N) \right|, \quad (37)$$

the right-hand-side (or the left-hand-side) inequality describes the half-space, defined by the hyperplane passing *outside* of the corresponding polytope via at least one of its vertices. A tight linear LHV inequality becomes an extreme one whenever this hyperplane describes a face of the corresponding polytope.

2.2. Constraints on joint probabilities

For an $S_1 \times \cdots \times S_N$-setting N-partite correlation, with at least $Q_n + 1$ (possibly, infinitely many) outcomes at each site, let us specify constraint (9) with functions[11]

$$\phi_n^{(s_n)}\left(\lambda_n^{(s_n)}\right) = \sum_{q_n = 1, \ldots, Q_n} \left\{ \tau_n^{(s_n, q_n)} \chi_{D_n^{(s_n, q_n)}}\left(\lambda_n^{(s_n)}\right) + z_n^{(s_n, q_n)} \right\}, \quad (38)$$

where $\tau_n^{(s_n, q_n)}$ and $z_n^{(s_n, q_n)}$ are arbitrary real numbers and $D_n^{(s_n, q_n)} \subset \Lambda_n^{(s_n)}$, $D_n^{(s_n, q_n)} \neq \varnothing$, $q_n \in \{1, \ldots, Q_n\}$, are any mutually disjoint subsets: $D_n^{(s_n, q_n)} \cap D_n^{(s_n, q_n')} = \varnothing$, $\forall q_n \neq q_n'$, observed under the s_nth measurement at the nth site and such that $\cup_{q_n} D_n^{(s_n, q_n)} \neq \Lambda_n^{(s_n)}$.

---

[11] Here, $\chi_D(\lambda)$, $\lambda \in \Lambda$, is an indicator function of a subset $D \subseteq \Lambda$, defined by relations: $\chi_D(\lambda) = 1$ if $\lambda \in D$ and $\chi_D(\lambda) = 0$ if $\lambda \notin D$.

Substituting these functions into the LHV constraint (9), making transformations similar to those in section 2.1 and renaming coefficients, we come to the following corollary of theorem 1.

Corollary 3. *Let an $S_1 \times \cdots \times S_N$-setting N-partite correlation experiment (1), satisfying the EPR locality[12] and with at least $(Q_n + 1)$ outcomes at each nth site, admit an LHV model. Then the tight linear LHV constraint on joint probabilities*

$$
\min_{\eta_1 \in \Xi_1, \ldots, \eta_N \in \Xi_N} \sum_{\substack{1 \leqslant n_1, \ldots, n_M \leqslant N, \\ M=1,\ldots,N}} F_M^{(\gamma)}(\eta_{n_1}, \ldots, \eta_{n_M})
$$

$$
\leqslant \sum_{\substack{1 \leqslant n_1 < \cdots < n_M \leqslant N, \\ M=1,\ldots,N}} \sum_{\substack{s_{n_1}, \ldots, s_{n_M}, \\ q_{n_1}, \ldots, q_{n_M}}} \gamma_{(s_{n_1}, \ldots, s_{n_M})}^{(q_{n_1}, \ldots, q_{n_M})} P_{(s_{n_1}, \ldots, s_{n_M})}\left(D_{n_1}^{(s_{n_1}, q_{n_1})} \times \cdots \times D_{n_M}^{(s_{n_M}, q_{n_M})}\right)
$$

$$
\leqslant \max_{\eta_1 \in \Xi_1, \ldots, \eta_N \in \Xi_N} \sum_{\substack{1 \leqslant n_1, \ldots, n_M \leqslant N, \\ M=1,\ldots,N}} F_M^{(\gamma)}(\eta_{n_1}, \ldots, \eta_{n_M}) \tag{39}
$$

holds for an arbitrary collection $\left\{\gamma_{(s_{n_1}, \ldots, s_{n_M})}^{(q_{n_1}, \ldots, q_{n_M})}\right\}$ of real coefficients and any events $D_n^{(s_n, q_n)} \subset \Lambda_n^{(s_n)}, D_n^{(s_n, q_n)} \neq \varnothing, q_n = 1, \ldots, Q_n$, observed under the s_nth measurement at the nth site, such that, for any $Q_n \geqslant 2$, these events are mutually incompatible: $D_n^{(s_n, q_n)} \cap D_n^{(s_n, q_n')} = \varnothing, \forall q_n \neq q_n'$, and satisfy the relation

$$
\cup_{q_n=1, \ldots, Q_n} D_n^{(s_n, q_n)} \neq \Lambda_n^{(s_n)}. \tag{40}
$$

In (39),

$$
F_M^{(\gamma)}(\eta_{n_1}, \ldots, \eta_{n_M}) = \sum_{\substack{s_{n_1}, \ldots, s_{n_M}, \\ q_{n_1}, \ldots, q_{n_M}}} \gamma_{(s_{n_1}, \ldots, s_{n_M})}^{(q_{n_1}, \ldots, q_{n_M})} \eta_{n_1}^{(s_{n_1}, q_{n_1})} \cdot \ldots \cdot \eta_{n_M}^{(s_{n_M}, q_{n_M})} \tag{41}
$$

is an M-linear form of real vectors $\eta_n \in \mathbb{R}^{S_n Q_n}$, with components $\eta_n^{(s_n, q_n)}$, and

$$
\Xi_n = \left\{ \eta_n \in \{0, 1\}^{S_n Q_n} \, \middle| \, \sum_{q_n=1, \ldots, Q_n} \eta_n^{(s_n, q_n)} \in \{0, 1\}, \forall s_n = 1, \ldots, S_n \right\}, \tag{42}
$$

for any $n = 1, \ldots, N$.

For an $S_1 \times S_2$-setting bipartite correlation experiment, the LHV constraint (39) takes the form

$$
\min_{\eta_1 \in \Xi_1, \eta_2 \in \Xi_2} \left\{ F_2^{(\gamma)}(\eta_1, \eta_2) + F_1^{(\gamma_1)}(\eta_1) + F_1^{(\gamma_2)}(\eta_2) \right\}
$$

$$
\leqslant \sum_{\substack{s_1, s_2, \\ q_1, q_2}} \gamma_{(s_1, s_2)}^{(q_1, q_2)} P_{(s_1, s_2)}\left(D_1^{(s_1, q_1)} \times D_2^{(s_2, q_2)}\right) + \sum_{s_1, q_1} \gamma_1^{(s_1, q_1)} P_1^{(s_1)}\left(D_1^{(s_1, q_1)}\right)
$$

$$
+ \sum_{s_2, q_2} \gamma_2^{(s_2, q_2)} P_2^{(s_2)}\left(D_2^{(s_2, q_2)}\right)
$$

$$
\leqslant \max_{\eta_1 \in \Xi_1, \eta_2 \in \Xi_2} \left\{ F_2^{(\gamma)}(\eta_1, \eta_2) + F_1^{(\gamma_1)}(\eta_1) + F_1^{(\gamma_2)}(\eta_2) \right\}, \tag{43}
$$

where: (i) $\gamma = \left(\gamma_{(s_1, s_2)}^{(q_1, q_2)}\right)$ is a real matrix of dimension $S_1 Q_1 \times S_2 Q_2$; (ii) $\gamma_1 \in \mathbb{R}^{S_1 Q_1}$, $\gamma_2 \in \mathbb{R}^{S_2 Q_2}$ are any real vectors with components $\gamma_1^{(s_1, q_1)}, \gamma_2^{(s_2, q_2)}$; (iii) $F_2^{(\gamma)}$ is a bilinear form and

---

[12] See condition (6) and notation (7).

$F_1^{(\gamma_1)}, F_1^{(\gamma_2)}$ are 1-linear forms, given by:

$$F_2^{(\gamma)}(\eta_1, \eta_2) = \sum_{\substack{s_1, s_2, \\ q_1, q_2}} \gamma_{(s_1, s_2)}^{(q_1, q_2)} \eta_1^{(s_1, q_1)} \eta_2^{(s_2, q_2)} = (\eta_1, \gamma \eta_2),$$

$$F_1^{(\gamma_1)}(\eta_1) = \sum_{s_1, q_1} \gamma_1^{(s_1, q_1)} \eta_1^{(s_1, q_1)} = (\eta_1, \gamma_1), \qquad (44)$$

$$F_1^{(\gamma_2)}(\eta_2) = \sum_{s_2, q_2} \gamma_2^{(s_2, q_2)} \eta_2^{(s_2, q_2)} = (\eta_2, \gamma_2).$$

Here, $(\cdot, \cdot)$ denotes the scalar product on the corresponding space $\mathbb{R}^{SQ}$.

Finally, let us specify the general form of tight LHV constraints on joint probabilities of *arbitrary* events, not necessarily of the product form. Taking in constraint (8) functions

$$\Psi_s\big(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}\big) = \sum_{q_s} \gamma_s^{(q_s)} \chi_{D_s^{(q_s)}}\big(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}\big), \qquad (45)$$

where $D_s^{(q_s)} \subseteq \Lambda_1^{(s_1)} \times \cdots \times \Lambda_N^{(s_N)}$, $q_s = 1, \ldots, Q_s$, are any events observed under a joint measurement $s := (s_1, \ldots, s_N)$, and $\chi_{D_s^{(q_s)}}\big(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}\big)$ is an indicator function[13] of a subset $D_s^{(q_s)}$, we derive the following corollary of theorem 1.

Corollary 4. *Let an $S_1 \times \cdots \times S_N$-setting N-partite correlation experiment (1) admit an LHV model. Then the tight linear LHV constraint on joint probabilities:*

$$\inf_{\lambda_1 \in \Lambda_1, \ldots, \lambda_N \in \Lambda_N} \sum_{q_s, s} \gamma_s^{(q_s)} \chi_{D_s^{(q_s)}}\big(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}\big) \leqslant \sum_{q_s, s} \gamma_s^{(q_s)} P_s\big(D_s^{(q_s)}\big)$$

$$\leqslant \sup_{\lambda_1 \in \Lambda_1, \ldots, \lambda_N \in \Lambda_N} \sum_{q_s, s} \gamma_s^{(q_s)} \chi_{D_s^{(q_s)}}\big(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}\big) \qquad (46)$$

holds for any real coefficients $\gamma_s^{(q_s)}$ and any events $D_s^{(q_s)} \subseteq \Lambda_1^{(s_1)} \times \cdots \times \Lambda_N^{(s_N)}$, $q_s = 1, \ldots, Q_s$, observed under an N-partite joint measurement $s := (s_1, \ldots, s_N)$ in family (1).

If, for example, we take in (46) coefficients, singling out only one joint measurement: $\gamma_{\widetilde{s}}^{(q_{\widetilde{s}})} = \delta_{s,\widetilde{s}}, \forall q_{\widetilde{s}}$, and events $D_s^{(q_s)} \subseteq \Lambda_1^{(s_1)} \times \cdots \times \Lambda_N^{(s_N)}$, that are incompatible and satisfy the relation $\cup_{q_s} D_s^{(q_s)} = \Lambda_1^{(s_1)} \times \cdots \times \Lambda_N^{(s_N)}$, then (46) reduces to the relation $\sum_{q_s} P_s\big(D_s^{(q_s)}\big) = 1$, fulfilled under any measurement.

3. Examples

The general representation (8) and its specifications in corollaries 1–4 incorporate as particular cases all Bell-type inequalities[14] for either correlation functions or joint probabilities that have been introduced in the literature.

In this section, we explicitly demonstrate this for the most known Bell-type inequalities. Namely, for: (1) the Clauser–Horne–Shimony–Holt (CHSH) inequality [2] for correlation functions; (2) the Clauser–Horne (CH) inequalities [3] for joint probabilities; (3) the Mermin–Klyshko (MK) inequality [6–8] for correlation functions; (4) the Bell-type inequalities for joint probabilities found computationally [17] by Collins and Gisin; (5) the Bell-type inequality for joint probabilities introduced recently by Zohren and Gill [18].

---

[13] See footnote 11.

[14] For the definition of a Bell-type inequality, see the beginning of Introduction.

Specifying constraint (46) for appropriate coefficients and events, it is also easy to derive all Bell-type inequalities derived by Collins, Gisin, Linden, Massar and Popescu in [13].

We stress that our approach allows us to derive all these inequalities in a *new unified manner* and also *to extend* the applicability ranges of even the well-known Bell-type inequalities.

3.1. The Clauser–Horne–Shimony–Holt (CHSH) inequality

For a 2×2-setting bipartite correlation experiment, with real-valued outcomes in $[-1, 1]$ of any spectral type, discrete or continuous, let us specify the tight LHV constraint (36) with coefficients $\gamma_{(s_1, s_2)}$ of the CHSH form [2]

$$\left(\gamma_{(s_1, s_2)}^{\text{CHSH}} \right) = \pm \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{47}$$

where the minus sign may equivalently stand in any matrix cell.

Note that, in a bipartite case, two parties are traditionally named as Alice and Bob and their measurements are usually specified by parameters a_i and b_k. Therefore, in the case of a *bipartite* correlation experiment, we further replace our general notations of section 2 for coefficients, outcomes and events by the following ones:

$$\gamma_{(s_1, s_2)}^{(q_1, q_2)} \to \gamma_{ik}^{(j,l)}, \quad \lambda_1^{(s_1)} \to \lambda_1^{(a_i)}, \quad \lambda_2^{(s_2)} \to \lambda_2^{(b_k)}, \qquad i = 1, \ldots, S_1, \quad k = 1, \ldots, S_2,$$
$$D_1^{(s_1, q_1)} \to A_i^{(j)}, \quad D_2^{(s_2, q_2)} \to B_k^{(l)}, \quad j = 1, \ldots, Q_1, \qquad l = 1, \ldots, Q_2. \tag{48}$$

Here, for concreteness, we refer site '1' to Alice and site '2' to Bob. For matrix $\gamma = \left(\gamma_{ik}^{(j,l)} \right) \equiv (\gamma_{ij,kl})$ of dimension $S_1 Q_1 \times S_2 Q_2$, the double indices (i, j) and (k, l) numerate, correspondingly, rows and columns in the order,

$$(1, 1), (1, 2) \ldots, (1, Q_1), \ldots, (S_1, 1), \ldots, (S_1, Q_1);$$
$$(1, 1), (1, 2) \ldots, (1, Q_2), \ldots, (S_2, 1), \ldots, (S_2, Q_2), \tag{49}$$

respectively, and the element $\gamma_{ik}^{(j,l)}$ stands in γ at the intersection of row (i, j) and column (k, l).

For the CHSH coefficients (47), the maximum of the absolute value of the bilinear form

$$F_2^{\text{CHSH}}(\eta_1, \eta_2) = \pm \left\{ \eta_1^{(1)} \eta_2^{(1)} + \eta_1^{(1)} \eta_2^{(2)} + \eta_1^{(2)} \eta_2^{(1)} - \eta_1^{(2)} \eta_2^{(2)} \right\} \tag{50}$$

over $\eta_1 = \left(\eta_1^{(1)}, \eta_1^{(2)} \right) \in \{-1, 1\}^2$, $\eta_2 = \left(\eta_2^{(1)}, \eta_2^{(2)} \right) \in \{-1, 1\}^2$ is equal to

$$\max_{(\eta_1, \eta_2) \in \{-1, 1\}^4} \left| F_2^{\text{CHSH}}(\eta_1, \eta_2) \right| = 2. \tag{51}$$

Substituting (51) into (36), we come to the following tight LHV constraint on correlation functions:

$$\left| \left\langle \lambda_1^{(a_1)} \lambda_2^{(b_1)} \right\rangle + \left\langle \lambda_1^{(a_1)} \lambda_2^{(b_2)} \right\rangle + \left\langle \lambda_1^{(a_2)} \lambda_2^{(b_1)} \right\rangle - \left\langle \lambda_1^{(a_2)} \lambda_2^{(b_2)} \right\rangle \right|_{\text{LHV}} \leqslant 2, \tag{52}$$

where minus sign may equivalently stand before any of four terms. This constraint holds for outcomes in $[-1, 1]$ of any spectral type, discrete or continuous, and constitutes the Clauser–Horne–Shimony–Holt (CHSH) inequality, derived originally in [2] for two ± 1-valued outcomes per site and further proved [5] by Bell to hold for any outcomes $\left| \lambda_1^{(a_i)} \right|, \left| \lambda_2^{(b_k)} \right| \leqslant 1, i, k = 1, 2$.

3.2. The Clauser–Horne (CH) inequalities

For a 2×2-setting bipartite correlation experiment, let us specify constraint (9) with the CHSH coefficients (47) and ± 1-valued functions

$$
\begin{aligned}
\phi_1^{(i)}\big(\lambda_1^{(a_i)}\big) &= 2\chi_{A_i}\big(\lambda_1^{(a_i)}\big) - 1, \qquad i = 1, 2, \\
\phi_2^{(k)}\big(\lambda_2^{(b_k)}\big) &= 2\chi_{B_k}\big(\lambda_2^{(b_k)}\big) - 1, \qquad k = 1, 2,
\end{aligned}
\tag{53}
$$

where $A_i \subseteq \Lambda_1^{(a_i)}$ and $B_k \subseteq \Lambda_2^{(s_2)}$ are any events observed by Alice and Bob under the corresponding measurements.

For these functions, the product expectations take the form,

$$
\begin{aligned}
\big\langle \phi_1^{(i)}\big(\lambda_1^{(a_i)}\big)\phi_2^{(k)}\big(\lambda_2^{(b_k)}\big)\big\rangle &= 1 + 4P_{(a_i,b_k)}(A_i \times B_k) - 2P_{(a_i,b_k)}\big(A_i \times \Lambda_2^{(b_k)}\big) \\
&\quad - 2P_{(a_i,b_k)}\big(\Lambda_1^{(a_i)} \times B_k\big),
\end{aligned}
\tag{54}
$$

and ranges (12) satisfy the relation: $\Phi_1, \Phi_2 \subseteq \{-1, 1\}^2$. The latter implies,

$$
\begin{aligned}
\max_{\xi_1 \in \Phi_1, \xi_2 \in \Phi_2} F_2^{\mathrm{CHSH}}(\xi_1, \xi_2) &\leqslant \max_{(\xi_1,\xi_2) \in \{-1,1\}^4} F_2^{\mathrm{CHSH}}(\xi_1, \xi_2), \\
\min_{\xi_1 \in \Phi_1, \xi_2 \in \Phi_2} F_2^{\mathrm{CHSH}}(\xi_1, \xi_2) &\geqslant \min_{(\xi_1,\xi_2) \in \{-1,1\}^4} F_2^{\mathrm{CHSH}}(\xi_1, \xi_2).
\end{aligned}
\tag{55}
$$

Taking into account (35), (51), we have:

$$
\begin{aligned}
\max_{(\xi_1,\xi_2) \in \{-1,1\}^4} F_2^{\mathrm{CHSH}}(\xi_1, \xi_2) &= -\min_{(\xi_1,\xi_2) \in \{-1,1\}^4} F_2^{\mathrm{CHSH}}(\xi_1, \xi_2) \\
&= \max_{(\xi_1,\xi_2) \in \{-1,1\}^4} \big|F_2^{\mathrm{CHSH}}(\xi_1, \xi_2)\big| \\
&= 2.
\end{aligned}
\tag{56}
$$

Substituting (54)–(56) into (9) and noting that, for *EPR local* measurements of Alice and Bob, the marginal probabilities in (54) have the form[15]

$$
P_{(a_i,b_k)}\big(A_i \times \Lambda_2^{(b_k)}\big) = P_1^{(a_i)}(A_i), \qquad P_{(a_i,b_k)}\big(\Lambda_1^{(a_i)} \times B_k\big) = P_2^{(b_k)}(B_k),
\tag{57}
$$

we come to the following LHV constraint on joint probabilities:

$$
\begin{aligned}
-1 \leqslant\ &P_{(a_1,b_1)}(A_1 \times B_1) + P_{(a_1,b_2)}(A_1 \times B_2) + P_{(a_2,b_1)}(A_2 \times B_1) \\
&- P_{(a_2,b_2)}(A_2 \times B_2) - P_1^{(a_1)}(A_1) - P_2^{(b_1)}(B_1) \leqslant 0.
\end{aligned}
\tag{58}
$$

This LHV constraint is valid for *any* events $A_i \subseteq \Lambda_1^{(a_i)}$, $B_k \subseteq \Lambda_2^{(b_k)}$, observed by Alice and Bob under measurements a_i, $i = 1, 2$, and b_k, $k = 1, 2$, respectively, and corresponds to the Clauser–Horne (CH) inequalities [3] on joint probabilities.

We stress that, in (58), outcome events may be *arbitrary*, in particular, certain: $A_i = \Lambda_1^{(a_i)}$, $B_k = \Lambda_2^{(b_k)}$, or impossible: $A_i = \varnothing$, $B_k = \varnothing$. This implies that, in the form (58), the CH inequalities incorporate *as particular cases* all positive probability relations considered in the literature[16] usually separately. If, for example, $A_2 = B_1 = \varnothing$ then (58) reduces to the positive probability relation $-1 \leqslant P_{(a_1,b_2)}(A_1 \times B_2) - P_1^{(a_1)}(A_1) \leqslant 0$, fulfilled under any bipartite joint measurement.

Note also that the CH inequalities (58) are equivalent[17] to the CHSH inequality (52) only in the case of two ± 1-valued outcomes at each site and the choice in (58) of uncertain possible events, say $A_i = \{1\}$, $B_k = \{1\}$, for any $i, k \in \{1, 2\}$.

---

[15] See condition (6) and notation (7).

[16] See, for example, in [4].

[17] In the sense that the validity of the CHSH inequality on correlation functions implies the validity of the CH inequalities on joint probabilities and vice versa.

3.3. The Mermin–Klyshko (MK) inequality

For a $\underbrace{2 \times \cdots \times 2}_{N}$-setting N-partite correlation experiment, with outcomes in $[-1, 1]$ of any spectral type, discrete or continuous, let us specify constraint (36) with coefficients $\gamma_{(s_1,\ldots,s_N)}$ defined by the recursion

$$\gamma_{(s_1,\ldots,s_{n-1},s_n)} = \gamma_{(s_1,\ldots,s_{n-1})} + (\delta_{s_n,1} - \delta_{s_n,2})\gamma_{(\overline{s_1},\ldots,\overline{s_{n-1}})}, \qquad 3 \leqslant n \leqslant N, \tag{59}$$

where $\left(\gamma_{(s_1,s_2)}\right) = \left(\gamma_{(s_1,s_2)}^{\text{CHSH}}\right) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\overline{s_n}$ is the element of set $\{1, 2\}\setminus\{s_n\}$.

In order to find the maximum of the absolute value of the N-linear form

$$F_N^{(\gamma)}(\eta_1, \ldots, \eta_N) = \sum_{s_1,\ldots,s_N=1,2} \gamma_{(s_1,\ldots,s_N)}\eta_1^{(s_1)} \cdot \cdots \cdot \eta_N^{(s_N)} \tag{60}$$

over vectors $\eta_1 \in \{-1, 1\}^2, \ldots, \eta_n \in \{-1, 1\}^2$, let us introduce n-linear forms, corresponding to the nth step in the recursion (59)

$$F_n^{(\gamma)}(\eta_1, \ldots, \eta_n) := \sum_{s_1,\ldots,s_n=1,2} \gamma_{(s_1,\ldots,s_n)}\eta_1^{(s_1)} \cdot \cdots \cdot \eta_n^{(s_n)},$$

$$\overline{F}_n^{(\gamma)}(\eta_1, \ldots, \eta_n) := \sum_{s_1,\ldots,s_n=1,2} \gamma_{(\overline{s_1},\ldots,\overline{s_n})}\eta_1^{(s_1)} \cdot \cdots \cdot \eta_n^{(s_n)}. \tag{61}$$

Substituting (59) into (61), we have

$$F_n^{(\gamma)}(\eta_1, \ldots, \eta_n) = \left(\eta_n^{(1)} + \eta_n^{(2)}\right)F_{n-1}^{(\gamma)}(\eta_1, \ldots, \eta_{n-1})$$
$$+ \left(\eta_n^{(1)} - \eta_n^{(2)}\right)\overline{F}_{n-1}^{(\gamma)}(\eta_1, \ldots, \eta_{n-1}), \qquad n \geqslant 3, \tag{62}$$

where

$$F_2^{(\gamma)}(\eta_1, \eta_2) = \eta_1^{(1)}\eta_2^{(1)} + \eta_1^{(1)}\eta_2^{(2)} + \eta_1^{(2)}\eta_2^{(1)} - \eta_1^{(2)}\eta_2^{(2)},$$

$$\overline{F}_2^{(\gamma)}(\eta_1, \eta_2) = -\eta_1^{(1)}\eta_2^{(1)} + \eta_1^{(1)}\eta_2^{(2)} + \eta_1^{(2)}\eta_2^{(1)} + \eta_1^{(2)}\eta_2^{(2)}. \tag{63}$$

Taking into account (51), (62), we prove by induction in n the following relation:

$$\max_{(\eta_1,\ldots,\eta_N)\in\{-1,1\}^{2N}} \left|F_N^{(\gamma)}(\eta_1, \ldots, \eta_N)\right| = 2^{N-1}, \quad N \geqslant 2. \tag{64}$$

Substituting (64) into (36), we come to the following $\underbrace{2 \times \cdots \times 2}_{N}$-setting tight LHV constraint on the full correlation functions:

$$\left| \sum_{s_1,\ldots,s_N\in\{1,2\}} \gamma_{(s_1,\ldots,s_N)}\left\langle\lambda_1^{(s_1)} \cdot \cdots \cdot \lambda_N^{(s_N)}\right\rangle_{\text{LHV}} \right| \leqslant 2^{N-1}, \tag{65}$$

where coefficients $\gamma_{(s_1,\ldots,s_N)}$ are given by (59). For $N = 2$, this inequality reduces to the CHSH inequality (52).

Let us now specify constraint (65) for a $\underbrace{2 \times \cdots \times 2}_{N}$-setting correlation experiment, with outcomes in $[-1, 1]$ of any spectral type, discrete or continuous, performed on a quantum state ρ on a complex separable Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, possibly infinite dimensional.

In the quantum case[18],

$$\left\langle\lambda_1^{(s_1)} \cdot \cdots \cdot \lambda_N^{(s_N)}\right\rangle_\rho = \int \lambda_1^{(s_1)} \cdot \cdots \cdot \lambda_N^{(s_N)}\text{tr}\left[\rho\left\{M_1^{(s_1)}\left(d\lambda_1^{(s_1)}\right) \otimes \cdots \otimes M_N^{(s_N)}\left(d\lambda_N^{(s_N)}\right)\right\}\right]$$

$$= \text{tr}\left[\rho\left(X_1^{(s_1)} \otimes \cdots \otimes X_N^{(s_N)}\right)\right], \tag{66}$$

---

[18] Here, $M_n^{(s_n)}(d\lambda_n^{(s_n)})$ is a positive operator-valued (POV) measure describing the s_nth measurement at the nth site, see, for example, section 3.1 in [20].

where

$$X_n^{(s_n)} = \int \lambda_n^{(s_n)} \mathrm{M}_n^{(s_n)}\big(\mathrm{d}\lambda_n^{(s_n)}\big) \tag{67}$$

is a bounded quantum observable on $\mathcal{H}_n$, observed under the s_nth measurement at the nth site and with the operator norm $\big\| X_n^{(s_n)} \big\| \leqslant 1$. If a Hilbert space $\mathcal{H}_n$, corresponding to the nth site, is infinite dimensional then observables $X_n^{(s_n)}$, $s_n = 1, 2$, may be of any spectral type, discrete or continuous.

From (66), (59) it follows that, in the quantum case,

$$\sum_{s_1,\dots,s_N} \gamma_{(s_1,\dots,s_N)} \big\langle \lambda_1^{(s_1)} \cdot \dots \cdot \lambda_N^{(s_N)} \big\rangle_\rho = \mathrm{tr}[\rho \mathcal{B}_N], \tag{68}$$

where $\mathcal{B}_N$ is the bounded quantum observable[19] on $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$, defined by the recursion

$$\begin{aligned}
\mathcal{B}_n &= \big(X_n^{(1)} + X_n^{(2)}\big) \otimes \mathcal{B}_{n-1} + \big(X_n^{(1)} - X_n^{(2)}\big) \otimes \widetilde{\mathcal{B}}_{n-1}, \qquad 2 \leqslant n \leqslant N, \\
\mathcal{B}_1 &= X_1^{(1)}, \qquad \widetilde{\mathcal{B}}_1 = X_1^{(2)},
\end{aligned} \tag{69}$$

where $\widetilde{\mathcal{B}}_n$ results from $\mathcal{B}_n$ by interchanging all $X_k^{(s_k)}$ to $X_k^{(\overline{s_k})}$, $s_k = 1, 2; k = 1, \dots n$.

Substituting (68) into (65), we come to the quantum version

$$|\mathrm{tr}[\rho \mathcal{B}_N]|_{\mathrm{LHV}} \leqslant 2^{N-1} \tag{70}$$

of the tight LHV constraint (65). By its form, this quantum LHV constraint coincides with the Mermin–Klyshko (MK) inequality, derived originally[20] [6–8] for the LHV description of spin measurements on N qubits and still discussed in the literature (see, for example, in [10]) only for a N-partite case with two dichotomic observables per site.

Our derivation of (70) shows that, for an N-partite quantum state ρ on $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$, possibly infinite dimensional, admitting the $\underbrace{2 \times \dots \times 2}_{N}$-setting LHV description[21], the MK inequality holds for *arbitrary* two quantum observables per site, not necessarily dichotomic. If $\mathcal{H}_n$ is infinite dimensional then the quantum observables measured at the nth site may be of any spectral type, discrete or continuous.

3.4. The Collins–Gisin inequalities

Let us now demonstrate that the tight LHV constraint (43) on joint probabilities incorporates as particular cases the *extreme* bipartite Bell-type inequalities found by Collins and Gisin [17] computationally. For short, we consider here the derivation of only two inequalities reported in [17].

For a 4×4-setting bipartite correlation experiment, with at least *two* outcomes per site, let us specify (43) with $Q_1 = Q_2 = 1$, matrix

$$\gamma = (\gamma_{ik}) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \tag{71}$$

---

[19] $\mathcal{B}_N$ represents a generalization of the so-called Bell operator for spin measurements on N qubits.

[20] Mermin's inequality [6] and the similar inequality of Ardehali [7] distinguish between even and odd values of N. For an odd N, the magnitude of the maximal violation of Mermin's inequality in a quantum case is higher than that of Ardehali. For an even N, the situation is opposite. Belinskii and Klyshko [8] proposed the single inequality, which is maximally violated, in comparison with those in [6, 7], for any N, even or odd. This inequality is usually referred to as the Mermin–Klyshko inequality.

[21] See section 5 of [20].

and vectors

$$\gamma_1 = (-1, 0, 0, 0), \qquad \gamma_2 = (-3, -2, -1, 0). \tag{72}$$

In this case, sets (42) take the form, $\Xi_1 = \Xi_2 = \{0, 1\}^4$, and maximum

$$\max_{\eta_1 \in \{0,1\}^4, \eta_2 \in \{0,1\}^4} \{(\eta_1, \gamma \eta_2) + (\eta_1, \gamma_1) + (\eta_2, \gamma_2)\} = 0, \tag{73}$$

achieved at, for example, $\eta_1 = (1, 1, 1, 1)$, $\eta_2 = (1, 1, 1, 1)$. Substituting (71)–(73) into the right-hand-side inequality of (43), we come to the tight LHV constraint:

$$\sum_{i,k} \gamma_{ik} P_{(a_i, b_k)}(A_i \times B_k) - P_1^{(a_1)}(A_1) - 3P_2^{(b_1)}(B_1) - 2P_2^{(b_2)}(B_2) - P_2^{(b_3)}(B_3) \leqslant 0, \tag{74}$$

corresponding to the extreme Bell-type inequality $I_{4422} \leqslant 0$, introduced in [17, equation (38)], and valid for any events: $A_i \subset \Lambda_1^{(a_i)}$, $A_i \neq \varnothing$, $B_k \subset \Lambda_2^{(b_k)}$, $B_k \neq \varnothing$, observed by Alice and Bob under the corresponding measurements.

For a 2×2-setting bipartite correlation experiment, with at least *three* outcomes per site, let us also specify (43) with $Q_1 = Q_2 = 2$, vectors

$$\gamma_1 = \gamma_2 = (-1, -1, 0, 0) \tag{75}$$

and matrix

$$\gamma = (\gamma_{ik}^{(j,l)}) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \tag{76}$$

where the element $\gamma_{ik}^{(j,l)}$ stands[22] in γ at the intersection of row (i, j) and column (k, l).

In this case, sets (42) are given by:

$$\Xi_1 = \left\{ \eta_1 \in \{0, 1\}^4 \,\middle|\, \sum_{j=1,2} \eta_1^{(i,j)} \in \{0, 1\}, i = 1, 2 \right\},$$
$$\Xi_2 = \left\{ \eta_2 \in \{0, 1\}^4 \,\middle|\, \sum_{l=1,2} \eta_2^{(k,l)} \in \{0, 1\}, k = 1, 2 \right\}, \tag{77}$$

and

$$\max_{\eta_1 \in \Xi_1, \eta_2 \in \Xi_2} \{(\eta_1, \gamma \eta_2) + (\eta_1, \gamma_1) + (\eta_2, \gamma_2)\} = 0, \tag{78}$$

achieved at, for example, $\eta_1 = (1, 0, 0, 1)$ and $\eta_2 = (1, 0, 0, 0)$.

Substituting (76)–(78) into the right-hand-side inequality of (43), we derive the tight LHV constraint

$$\sum_{i,j,k,l} \gamma_{ik}^{(j,l)} P_{(a_i, b_k)}\big(A_i^{(j)} \times B_k^{(l)}\big) - P_1^{(a_1)}\big(A_1^{(1)}\big) - P_1^{(a_1)}\big(A_1^{(2)}\big) - P_2^{(b_1)}\big(B_1^{(1)}\big) - P_2^{(b_1)}\big(B_1^{(2)}\big) \leqslant 0,$$

$$\tag{79}$$

corresponding to the extreme Bell-type inequality $I_{2233} \leqslant 0$, introduced analytically in [13, 14] and further confirmed computationally in [17, equation (39)]. This inequality is valid for any two incompatible events

$$A_i^{(j)} \subset \Lambda_1^{(a_i)}, \quad A_i^{(j)} \neq \varnothing, \quad j = 1, 2, \quad A_i^{(1)} \cap A_i^{(2)} = \varnothing, \quad A_i^{(1)} \cup A_i^{(2)} \neq \Lambda_1^{(a_i)}, \tag{80}$$

observed by Alice under measurement a_i, $i = 1, 2$, and any two incompatible events

$$B_k^{(l)} \subseteq \Lambda_2^{(b_k)}, \quad B_k^{(l)} \neq \varnothing, \quad l = 1, 2, \quad B_k^{(1)} \cap B_k^{(2)} = \varnothing, \quad B_k^{(1)} \cup B_k^{(2)} \neq \Lambda_2^{(b_k)}, \tag{81}$$

observed by Bob under measurement b_k, $k = 1, 2$.

---

[22] See also (49).

3.5. The Zohren–Gill inequality

Finally, consider a 2×2-setting bipartite correlation experiment with K real-valued outcomes per site: $\lambda_1^{(s_1)}, \lambda_2^{(s_2)} \in \Lambda = \{1, \ldots, K\}$, where $2 \leqslant K \leqslant \infty$.

For this case, let us specify the tight linear LHV constraint (46) with $\gamma_{(s_1, s_2)} = 1$, $q_{(s_1, s_2)} = 1$, $\forall s_1, s_2 \in \{1, 2\}$, and events:

$$
\begin{aligned}
D_{(s_1, s_2)} &= \left\{\lambda_2^{(s_2)} > \lambda_1^{(s_1)}\right\} \subset \Lambda \times \Lambda, \qquad \text{if} \quad s_1 = s_2 \in \{1, 2\}, \\
D_{(s_1, s_2)} &= \left\{\lambda_1^{(s_1)} > \lambda_2^{(s_1)}\right\} \subset \Lambda \times \Lambda, \qquad \text{if} \quad s_1 \neq s_2 \in \{1, 2\}.
\end{aligned}
\tag{82}
$$

We have

$$
\begin{aligned}
\chi_{D_{(s_1, s_2)}}\left(\lambda_1^{(s_1)}, \lambda_2^{(s_2)}\right) &= \theta\left(\lambda_2^{(s_2)} - \lambda_1^{(s_1)}\right), \qquad \text{if} \quad s_1 = s_2 \in \{1, 2\}, \\
\chi_{D_{(s_1, s_2)}}\left(\lambda_1^{(s_1)}, \lambda_2^{(s_2)}\right) &= \theta\left(\lambda_1^{(s_1)} - \lambda_2^{(s_2)}\right), \qquad \text{if} \quad s_1 \neq s_2 \in \{1, 2\},
\end{aligned}
\tag{83}
$$

where $\theta(x - y) = 1$, for $x > y$, and $\theta(x - y) = 0$, for $x \leqslant y$. Substituting (83) into the left-hand-side inequality of the LHV constraint (46), we derive the following expression for

$$
\inf \left\{\theta\left(\lambda_2^{(1)} - \lambda_1^{(1)}\right) + \theta\left(\lambda_1^{(1)} - \lambda_2^{(2)}\right) + \theta\left(\lambda_2^{(2)} - \lambda_1^{(2)}\right) + \theta\left(\lambda_1^{(2)} - \lambda_2^{(1)}\right)\right\} = 1
\tag{84}
$$

over all $\lambda_1^{(1)}, \lambda_1^{(2)}, \lambda_2^{(1)}, \lambda_2^{(2)} \in \{1, \ldots, K\}$, and, therefore, the following tight LHV constraint,

$$
\begin{aligned}
P_{(1,1)}\left(\left\{\lambda_2^{(1)} > \lambda_1^{(1)}\right\}\right) &+ P_{(1,2)}\left(\left\{\lambda_1^{(1)} > \lambda_2^{(2)}\right\}\right) \\
&+ P_{(2,2)}\left(\left\{\lambda_2^{(2)} > \lambda_1^{(2)}\right\}\right) + P_{(2,1)}\left(\left\{\lambda_1^{(2)} > \lambda_2^{(1)}\right\}\right) \geqslant 1,
\end{aligned}
\tag{85}
$$

which is valid for any number K of outcomes per site, in particular, for inifinitely many outcomes ($K = \infty$) at each site. This tight LHV constraint constitutes the Bell-type inequality derived quite differently in [18].

4. Conclusions

In the present paper, which is a sequel to [20], we have introduced in rigorous mathematical terms a single general representation for all tight linear LHV constraints arising under an $S_1 \times \cdots \times S_N$-setting N-partite correlation experiment with outcomes of any spectral type, discrete or continuous. For correlation functions and joint probabilities, this representation is formulated in terms of multilinear forms and this allows us:

- to prove in a general setting that the form of any correlation Bell-type inequality does not depend on a spectral type of outcomes at different sites, in particular, on their numbers and is determined only by extremal values of outcomes at each site,
- to specify the general form of bounds in Bell-type inequalities for joint probabilities;
- to present the new concise proofs for all the most known Bell-type inequalities introduced in the literature ever since the seminal publication of Bell [1] and also to extend the applicability ranges of some of these inequalities.

Note that the LHV constraints, reproduced in sections 3.1–3.4, are not only tight, but, as is proved in [4, 10, 17], respectively, each of these inequalities is extreme for the corresponding setting of a correlation experiment. However, for an arbitrary multipartite case, there does not still exist an effective general way to single out extreme Bell-type inequalities. Though the polytope approach is very useful from the descriptive-geometrical point of view, there is not much sense of finding of extreme Bell-type inequalities by listing of a huge number of faces of a highly dimensional polytope whereas many of these faces correspond to trivial probabilistic constraints while others can be subdivided into only a few classes different by their form.

The approach, introduced in the present paper, is based on general properties of multilinear forms and this points to a possibility of a new direction in finding extreme Bell-type inequalities for an arbitrary multipartite case. This problem will be analysed in our further publications.

Appendix

Proof of lemma 1. For the real-valued function

$$W(\eta) := \sum_{\substack{1 \leqslant n_1 < \cdots < n_M \leqslant N, \\ M=1,\ldots,N}} F_M^{(\gamma)}(\eta_{n_1}, \ldots, \eta_{n_M}) \tag{A.1}$$

continuous on $\mathbb{R}^{S_1+\cdots+S_N}$, its supremum and infimum over $\eta = (\eta_1, \ldots, \eta_N) \in \Lambda_1 \times \cdots \times \Lambda_N \subseteq [-1, 1]^{S_1+\cdots+S_N}$ have the form:

$$\sup_{\eta \in \Lambda_1 \times \cdots \times \Lambda_N} W(\eta) = \sup_{\eta \in \overline{\Lambda}_1 \times \cdots \times \overline{\Lambda}_N} W(\eta), \qquad \inf_{\eta \in \Lambda_1 \times \cdots \times \Lambda_N} W(\eta) = \inf_{\eta \in \overline{\Lambda}_1 \times \cdots \times \overline{\Lambda}_N} W(\eta), \tag{A.2}$$

where

$$\sup_{\eta \in \overline{\Lambda}_1 \times \cdots \times \overline{\Lambda}_N} W(\eta) = \max_{\eta \in \overline{\Lambda}_1 \times \cdots \times \overline{\Lambda}_N} W(\eta), \qquad \inf_{\eta \in \overline{\Lambda}_1 \times \cdots \times \overline{\Lambda}_N} W(\eta) = \min_{\eta \in \overline{\Lambda}_1 \times \cdots \times \overline{\Lambda}_N} W(\eta). \tag{A.3}$$

Therefore,

$$\sup_{\eta \in \Lambda_1 \times \cdots \times \Lambda_N} W(\eta) = \max_{\eta \in \overline{\Lambda}_1 \times \cdots \times \overline{\Lambda}_N} W(\eta), \qquad \inf_{\eta \in \Lambda_1 \times \cdots \times \Lambda_N} W(\eta) = \min_{\eta \in \overline{\Lambda}_1 \times \cdots \times \overline{\Lambda}_N} W(\eta). \tag{A.4}$$

From relation (28) it follows:

$$\begin{aligned}
\max_{\eta \in \{-1,1\}^d} W(\eta) &\leqslant \max_{\eta \in \overline{\Lambda}_1 \times \cdots \times \overline{\Lambda}_N} W(\eta) \leqslant \max_{\eta \in [-1,1]^d} W(\eta), \\
\min_{\eta \in [-1,1]^d} W(\eta) &\leqslant \min_{\eta \in \overline{\Lambda}_1 \times \cdots \times \overline{\Lambda}_N} W(\eta) \leqslant \min_{\eta \in \{-1,1\}^d} W(\eta),
\end{aligned} \tag{A.5}$$

where $d = S_1 + \cdots + S_N$. Note that $\eta = (\eta_1^{(1)}, \ldots, \eta_1^{(S_1)}, \ldots, \eta_N^{(1)}, \ldots, \eta_N^{(S_N)}) \in \mathbb{R}^d$ and function $W(\eta)$ is twice continuously differentiable on $\mathbb{R}^d$ with the second partial derivatives

$$\frac{\partial^2 W(\eta)}{\partial (\eta_n^{(s_n)})^2} = 0. \tag{A.6}$$

Therefore, function $W(\eta)$, $\eta \in \mathbb{R}^d$, is harmonic[23]. From the maximum principle for harmonic functions it follows that the maximum and the minimum of function $W(\eta)$ in the hypercube $\mathbb{V}_d := [-1, 1]^d \subset \mathbb{R}^d$ are reached on boundary Γ_d of $\mathbb{V}_d$, that is:

$$\max_{\eta \in [-1,1]^d} W(\eta) = \max_{\eta \in \Gamma_d} W(\eta), \qquad \min_{\eta \in [-1,1]^d} W(\eta) = \min_{\eta \in \Gamma_d} W(\eta). \tag{A.7}$$

Since the boundary Γ_d of $\mathbb{V}_d$ represents the union of $(d-1)$-dimensional hypercubes $\mathbb{V}_{d-1}^{(k)}$, $k = 1, \ldots, 2d$, the right-hand sides of relations (A.7) are given by

$$\max_{\eta \in \Gamma_d} W(\eta) = \max_{k=1,\ldots,2d} \{ \max_{\eta \in \mathbb{V}_{d-1}^{(k)}} W(\eta) \}, \qquad \min_{\eta \in \Gamma_d} W(\eta) = \min_{k=1,\ldots,2d} \{ \min_{\eta \in \mathbb{V}_{d-1}^{(k)}} W(\eta) \}. \tag{A.8}$$

Further, on each $(d-1)$-dimensional hypercube $\mathbb{V}_{d-1}^{(k)}$, function $W(\eta)|_{\mathbb{V}_{d-1}^{(k)}}$, depending on $(d-1)$ components of η, is harmonic and, therefore, reaches its maximum (minimum) on boundary $\Gamma_{d-1}^{(k)}$ of $\mathbb{V}_{d-1}^{(k)}$. The latter, in turn, consists of $(d-2)$-dimensional hypercubes $\mathbb{V}_{d-2}^{(m)}$.

---

[23] On this notion, see any textbook on equations of mathematical physics.

Since, in total, the boundary Γ_d contains $4d(d-1)$ of $(d-2)$-dimensional hypercubes $\mathbb{V}_{d-2}^{(m)}$, relation (A.8) reduces to:

$$
\begin{aligned}
\max_{\eta \in \Gamma_d} W(\eta) &= \max_{k=1,\dots,2d} \{ \max_{\eta \in \mathbb{V}_{d-1}^{(k)}} W(\eta) \} \\
&= \max_{m=1,\dots,4d(d-1)} \{ \max_{\eta \in \mathbb{V}_{d-2}^{(m)}} W(\eta) \},
\end{aligned}
\tag{A.9}
$$

with a similar relation for minimum.

Recall that the number of l-dimensional hypercubes on the boundary Γ_d is equal to

$$
\frac{d!}{(d-l)!} 2^{d-l},
\tag{A.10}
$$

in particular, $d \cdot 2^{d-1}$ edges ('1'-dimensional hypercubes) and 2^d vertices ('0'-dimensional hypercubes).

Continuing to reduce the dimension of hypercubes in formula (A.9), we finally come to the maximum (minimum) over all '0'-dimensional hypercubes, that is, over set $\{-1, 1\}^d$ of all 2^d vertices of hypercube $\mathbb{V}_d = [-1, 1]^d$. Thus

$$
\max_{\eta \in [-1,1]^d} W(\eta) = \max_{\eta \in \{-1,1\}^d} W(\eta), \qquad \min_{\eta \in [-1,1]^d} W(\eta) = \min_{\eta \in \{-1,1\}^d} W(\eta).
\tag{A.11}
$$

From (A.4), (A.5) and (A.11) it follows:

$$
\sup_{\eta \in \Lambda_1 \times \cdots \times \Lambda_N} W(\eta) = \max_{\eta \in \{-1,1\}^d} W(\eta), \qquad \inf_{\eta \in \Lambda_1 \times \cdots \times \Lambda_N} W(\eta) = \min_{\eta \in \{-1,1\}^d} W(\eta).
\tag{A.12}
$$

This proves the statement of lemma 1. $\qquad\qquad\square$

References

[1] Bell J S 1964 *Physics* **1** 195
[2] Clauser J F, Horne M A, Shimony A and Holt R A 1969 *Phys. Rev. Lett.* **23** 880
[3] Clauser J F and Horne M A 1974 *Phys. Rev.* D **10** 526
[4] Fine A 1982 *Phys. Rev. Lett.* **48** 291
[5] Gard A and Mermin N D 1982 *Phys. Rev. Lett.* **49** 1220
[6] Mermin N D 1990 *Phys. Rev. Lett.* **65** 1838
[7] Ardehali M 1992 *Phys. Rev.* A **46** 5375
[8] Belinskii A V and Klyshko D N 1993 *Sov. Phys.—Usp.* **36** 653
[9] Pitowsky I and Swozil K 2001 *Phys. Rev.* A **64** 014102
[10] Werner R F and Wolf M M 2001 *Phys. Rev.* A **64** 032112
[11] Weinfurter R F and Zukowski M 2001 *Phys. Rev.* A **64** 010102 (R)
[12] Zukowski M and Brukner C 2002 *Phys. Rev. Lett.* **88** 210401
[13] Collins D, Gisin N, Linden N, Massar S and Popescu S 2002 *Phys. Rev. Lett.* **88** 040404
[14] Kaszlikowski D, Kwek L C, Chen J-L, Zukowski M and Oh C H 2002 *Phys. Rev.* A **65** 032118
[15] Masanes L 2002 *Quantum Inf. Comput.* **3** 345
[16] Sliwa C 2003 arXiv:quant-ph/0305190
[17] Collins D and Gisin N 2004 *J. Phys. A: Math. Gen.* **37** 1775
[18] Zohren S and Gill R D 2008 *Phys. Rev. Lett.* **100** 120406
[19] Pitowsky I 1989 *Quantum Probability—Quantum Logic* (Berlin: Springer)
[20] Loubenets E R 2008 *J. Phys. A: Math. Theor.* **41** 415303
[21] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777